

Stability of Gieseker stable sheaves on K3 surfaces in the sense of Bridgeland and some applications

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Abstract

We show that some Gieseker stable sheaves on a projective K3 surface X are stable with respect to a stability condition of Bridgeland on the derived category of X if the stability condition is in explicit subsets of the space of stability conditions depending on the sheaves. Furthermore we shall give two applications of the result. As a part of these applications, we show that the fine moduli space of Gieseker stable torsion free sheaves on a K3 surface with Picard number one is the moduli space of μ -stable locally free sheaves if the rank of the sheaves is not a square number.

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1 Introduction

In the article [1], Bridgeland constructed the theory of stability conditions on arbitrary triangulated categories \mathcal{D} . A stability condition σ is a pair (\mathcal{A}, Z) with some axioms where \mathcal{A} is the heart of a bounded t -structure on \mathcal{D} and Z is a group homomorphism from the Grothendieck group $K(\mathcal{D})$ of \mathcal{D} to \mathbb{C} . Let $\text{Stab}(\mathcal{D})$ be the space of stability conditions on \mathcal{D} . If $\text{Stab}(\mathcal{D})$ is not empty then $\text{Stab}(\mathcal{D})$ is known to be a complex manifold by [1]. If a stability condition σ on \mathcal{D} exists we can define the notion of σ -stability for objects $E \in \mathcal{D}$.

Suppose that \mathcal{D} is the bounded derived category $D(X)$ of coherent sheaves on a smooth projective variety X over \mathbb{C} . In this paper we study the case where X is a projective K3 surface. Then as is well-known, the space $\text{Stab}(X)$ of stability conditions on $D(X)$ is not empty by virtue of Bridgeland [2]. Then for coherent sheaves on X we have the notion of σ -stability in addition to Gieseker stability and μ -stability. Thus it is natural to compare these stabilities. We shall give an partial answer to this problem.

We have two goals. The first goal is to show the σ -stability of Gieseker stable (or μ -stable) sheaves on X if σ is in explicit subsets of $\text{Stab}(X)$ depending on the sheaves. This result will be proved in Theorems 4.4 and 4.10. The second goal is to give two applications of these two theorems.

We comment on Theorems 4.4 and 4.10. Recall that the space $\text{Stab}(X)$ has the subset $U(X)$ described by

$$U(X) = \{ \sigma \in \text{Stab}(X) | \forall x \in X, \mathcal{O}_x \text{ is } \sigma\text{-stable with a common phase and } \sigma \text{ is good, locally finite and numerical} \}, \quad (1.1)$$

where \mathcal{O}_x is the structure sheaf of a closed point $x \in X$. Very roughly this subset $U(X)$ is also a trivial $\tilde{GL}^+(2, \mathbb{R})$ bundle over a set $V(X)$, where $\tilde{GL}^+(2, \mathbb{R})$ is the universal cover of $GL^+(2, \mathbb{R})$ (See also Section 3). In addition $V(X)$ is roughly parametrized by \mathbb{R} divisors β and \mathbb{R} ample divisors ω . Hence we can write $\sigma \in U(X)$ as $\sigma = \sigma_{(\beta, \omega)} \cdot \tilde{g}$ where $\sigma_{(\beta, \omega)} \in V(X)$ and $\tilde{g} \in \tilde{GL}^+(2, \mathbb{R})$. It is shown in [2] that if we take a sufficiently large $\omega \gg 0$, then the notion of σ -stability is just (β, ω) -twisted stability for coherent sheaves. Namely, for any sufficiently large $\lambda \gg 0$ if $E \in D(X)$ is $\sigma_{(\beta, \lambda\omega)} \cdot \tilde{g}$ -stable then E is a (β, ω) -twisted stable sheaf and vice versa. In some sense we strengthen this result. We give an explicit bound for λ depending on sheaves so that Gieseker stable sheaf is $\sigma_{(\beta, \lambda\omega)} \cdot \tilde{g}$ -stable.

In Theorem 5.4, which is the first application, we study fine moduli spaces of Gieseker stable sheaves on a projective K3 surface X with Picard number one. More precisely we show the fine moduli space of Gieseker stable torsion free sheaves is the fine moduli space of μ -stable locally free sheaves if the rank of the sheaves is not a square number. We also show that if the rank is a square number then the fine moduli space is the moduli space of μ -stable locally free sheaves or the moduli space of properly Gieseker stable torsion free sheaves¹. Furthermore we show that if the latter case occurs then the moduli space is isomorphic to X itself. The key idea of the proof of Theorem 5.4 is to compare two Jordan-Hölder filtrations of a Gieseker stable sheaf with respect to μ -stability and σ -stability for some $\sigma \in \text{Stab}(X)$. This comparison is enabled by Proposition 5.2. In this proposition we show that the σ -stability of some Gieseker stable sheaf E on X is equivalent to the μ stability and the local freeness of E if σ is in a subset $V_{v(E)}^-$ (See Section 5 for the definition of $V_{v(E)}^-$). As a consequence of Theorem 5.4, we see that any non trivial Fourier-Mukai partner should be the fine moduli of μ -stable locally free sheaf.

The second application is Theorem 6.7, which is a generalization of [8, Theorem 1.1]. Let $\Phi : D(Y) \rightarrow D(X)$ be an equivalence where X and Y are projective K3 surfaces with Picard number one and let $\Phi_* : \text{Stab}(Y) \rightarrow \text{Stab}(X)$ be a natural map induced by Φ . In [8, Theorem 1.1], the author showed that, if Φ satisfies the condition $\Phi_*U(Y) = U(X)$ then the equivalence Φ is given by $M \otimes f_*(-)[n]$ where M is a line bundle on X , f is an isomorphism from Y to X and $n \in \mathbb{Z}$.

As the second application, we remove the assumption that the Picard number of X is one from [8, Theorem 1.1]. We proceed as follows. For an equivalence $\Phi : D(Y) \rightarrow D(X)$ satisfying the assumption $\Phi_*U(Y) = U(X)$, one can see that it is enough to prove $\Phi(\mathcal{O}_y) = \mathcal{O}_x[n]$ where $x \in X$ and $n \in \mathbb{Z}$, since $U(X)$ is given by (1.1). In [8], this was proved by using [8, Theorem 6.6]. Hence the crucial part of the proof of [8, Theorem 1.1] is [8, Theorem 6.6]. A necessary generalization of this result of [8] will be done in Corollary 6.6 by applying Theorem 4.6.

We finally explain the motivation of our study. In the previous paper [8] we also showed the σ -stability of Gieseker stable or μ -stable sheaves. Before we started the previous study we expected that there would be a Gieseker stable torsion free sheaf E with $\dim \text{Ext}_X^1(E, E) = 2$ on a polarized K3 surface (X, L) such that E is σ -stable for all σ in $U(X)$. This conjecture is based on the fact that any line bundles P with $c_1(P) = 0$ on an abelian

¹Namely the sheaf is neither μ -stable nor locally free.

surface² are σ -stable for all $\sigma \in U(X)$. However throughout the previous study we showed our conjecture never holds if X is a projective K3 surface with Picard number one. Hence we had to give up our first conjecture and tried the following two things. One is to find explicit subsets of $U(X)$ depending on Gieseker stable sheaves so that the sheaves are σ -stable if σ is in the subsets. The other is to find interesting applications of σ -stability of Gieseker stable sheaves.

2 Review of classical stability for sheaves

In this section we recall the μ -stability, Gieseker stability and twisted stability for coherent sheaves on a projective K3 surface.

We first introduce some notations. Throughout this section X is a projective K3 surface over \mathbb{C} . Let A and B be in $D(X)$. If the i -th cohomology $H^i(A)$ is concentrated only at degree $i = 0$, we call A a *sheaf*. We put $\mathrm{Hom}_X^n(A, B) := \mathrm{Hom}_{D(X)}(A, B[n])$ and $\mathrm{hom}_X^n(A, B) := \dim_{\mathbb{C}} \mathrm{Hom}_X^n(A, B)$ where $[n]$ means $n \in \mathbb{Z}$ times shifts. We remark that

$$\mathrm{Hom}_X^n(A, B) = \mathrm{Hom}_X^{2-n}(B, A)^*$$

by the Serre duality. Then the Euler pairing $\chi(E, F) = \sum_i (-1)^i \mathrm{hom}_X^i(E, F)$ is a \mathbb{Z} -bilinear symmetric form on the Grothendieck group $K(X)$ of $D(X)$.

Let $\mathcal{N}(X)$ be the quotient of $K(X)$ by numerical equivalence with respect to the Euler pairing χ . Then $\mathcal{N}(X)$ is isomorphic to

$$H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z})$$

where $\mathrm{NS}(X)$ is the Néron-Severi Lattice of X . For $E \in D(X)$, we define the Mukai vector $v(E)$ of E by $ch(E)\sqrt{td_X}$. Then $v(E) = r_E \oplus \delta_E \oplus s_E$ is in $H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z})$ and we have $r_E = \mathrm{rank} E$, $\delta_E = c_1(E)$ and $s_E = \chi(\mathcal{O}_X, E) - r_E$.

Let $\langle -, - \rangle$ be the Mukai pairing on $\mathcal{N}(X)$:

$$\langle r \oplus \delta \oplus s, r' \oplus \delta' \oplus s' \rangle = \delta\delta' - rs' - r's,$$

where $r \oplus \delta \oplus s, r' \oplus \delta' \oplus s' \in \mathcal{N}(X)$. Then, by the Riemann-Roch formula, we see

$$\chi(E, F) = -\langle v(E), v(F) \rangle.$$

We secondly recall the notion of the μ -stability. For a torsion free sheaf F and an ample divisor ω , the *slope* $\mu_\omega(F)$ is defined by $(c_1(F) \cdot \omega) / \mathrm{rank} F$.

²These line bundles are Gieseker stable with $\dim \mathrm{Ext}_X^1(E, E) = 2$.

If the inequality $\mu_\omega(A) \leq \mu_\omega(F)$ holds for any non-trivial subsheaf A of F , then F is said to be μ_ω -semistable. Moreover if the strict inequality $\mu_\omega(A) < \mu_\omega(F)$ holds for any non-trivial subsheaf A with $\text{rank } A < \text{rank } F$, then F is said to be μ_ω -stable. If $\text{NS}(X) = \mathbb{Z}L$, we write μ -(semi)stable instead of μ_L -(semi)stable. The notion of the μ_ω -stability admits the Harder-Narashimhan filtration of F (details in [3]). We define $\mu_\omega^+(F)$ by the maximal slope of semistable factors of F , and $\mu_\omega^-(F)$ by the minimal slope of semistable factors of F .

Let β be an \mathbb{R} divisor and ω an \mathbb{R} ample divisor on X^3 . For a pair (β, ω) we define the notion of (β, ω) twisted stability introduced by [7]. For a torsion free sheaf E with $v(E) = r_E \oplus \delta_E \oplus s_E$, we define a polynomial $p_{(\beta, \omega)}(E)$ by

$$p_{(\beta, \omega)}(E) := \frac{\omega^2}{2} \cdot n^2 + \left(\frac{\delta_E}{r_E} - \beta \right) \omega \cdot n + \frac{s_E}{r_E} - \frac{\delta_E \beta}{r_E} + \frac{\beta^2}{2} + 1 \in \mathbb{R}[n].$$

Suppose that ω is an integral class and put $\omega = \mathcal{O}_X(1)$. Then $p_{(\beta, \omega)}(n)$ is simply given by

$$p_{(\beta, \omega)}(E) = - \frac{\langle v(\mathcal{O}_X(-n)), \exp(-\beta)v(E) \rangle}{r_E}.$$

Definition 2.1. Let E be a torsion free sheaf on a projective K3 surface X . E is said to be (β, ω) -twisted (semi)stable if $p_{(\beta, \omega)}(F) < (\leq) p_{(\beta, \omega)}(E)$ for any nontrivial subsheaf F of E .

Moreover if $\beta = 0$ then E is said to be Gieseker (semi)stable with respect to ω . For a torsion free sheaf E , we write $p_\omega(E)$ instead of $p_{(0, \omega)}(E)$.

Remark 2.2. For a torsion free sheaf E , we can easily check the following relation between the μ_ω -stability and the (β, ω) -twisted stability:

$$\mu_\omega\text{-stable} \Rightarrow (\beta, \omega)\text{-twisted stable} \Rightarrow (\beta, \omega)\text{-twisted semistable} \Rightarrow \mu_\omega\text{-semistable}.$$

We also see the following relation between the μ_ω -stability and the Gieseker stability:

$$\mu_\omega\text{-stable} \Rightarrow \text{Gieseker stable} \Rightarrow \text{Gieseker semistable} \Rightarrow \mu_\omega\text{-semistable}.$$

Finally we cite the following lemma which plays an important role when we study the space of stability conditions on abelian or K3 surfaces. A prototype of Lemma 2.3 was first proved by Mukai and Bridgeland. Finally [4] refined it.

³Originally the notion of twisted stability is defined on projective surfaces. To avoid the complexity we add the assumption that X is a projective K3 surface.

Lemma 2.3. ([4, Lemma 2.7]) *Let X be an abelian surface or a K3 surface. Suppose that $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle in $D(X)$. If $\mathrm{hom}_X^i(A, C) = 0$ for any $i \leq 0$ and $\mathrm{hom}_X^j(C, C) = 0$ for any $j < 0$ then we have the following inequality:*

$$0 \leq \mathrm{hom}_X^1(A, A) + \mathrm{hom}_X^1(C, C) \leq \mathrm{hom}_X^1(B, B).$$

3 Review of Bridgeland's work

In this section we briefly recall the theory of stability conditions. The details are in the original articles [1] and [2]. For a projective K3 surface X we put $\mathrm{NS}(X)_{\mathbb{R}} = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathrm{Amp}(X)$ by the set of \mathbb{R} -ample divisors.

Let \mathcal{A} be the heart of a bounded t -structure on the derived category $D(X)$ of X and let Z be a group homomorphism from $K(X)$ to \mathbb{C} . Notice that $K(X)$ is isomorphic to the Grothendieck of the heart \mathcal{A} . The morphism Z is called a *stability function* on \mathcal{A} if Z satisfies the following:

$$0 \neq E \in \mathcal{A} \Rightarrow Z(E) = m \exp(\sqrt{-1}\pi\phi_E),$$

where $m \in \mathbb{R}_{>0}$ and ϕ_E is in the interval $(0, 1]$. Then we put $\arg Z(E) = \phi_E$ and call the pair (\mathcal{A}, Z) a *stability pair* on $D(X)$. If we take a stability pair (\mathcal{A}, Z) , we can define the notion of Z -stability for objects in \mathcal{A} as follows:

Definition 3.1. Let (\mathcal{A}, Z) be a stability pair on $D(X)$ and E in \mathcal{A} . The object E is said to be Z -(semi)stable if E satisfies $\arg Z(F) < (\leq) \arg(E)$ for any non-trivial subobject.

By using the notion of Z -stability, we define a stability condition on $D(X)$ as follows:

Definition 3.2. A stability pair (\mathcal{A}, Z) is said to be a *stability condition* on $D(X)$ if any $E \in \mathcal{A}$ has the filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$ such that $A_i := E_i/E_{i-1}$ ($i = 1, 2, \dots, n$) is Z -semistable with $\arg Z(A_1) > \cdots > \arg Z(A_n)$. We call such a filtration the *Harder-Narashimhan filtration* of E . Moreover if Z factors through $\mathcal{N}(X)$, σ is said to be *numerical*.

Let $\sigma = (\mathcal{A}, Z)$ be a stability condition on $D(X)$. Then we can define the notion of σ -stability for any object in $D(X)$ ⁴. An object $E \in D(X)$ is

⁴For a stability pair (\mathcal{A}, Z) , we can logically define the notion of σ -stability for objects in $D(X)$. However in the original article [1], the notion of stability of arbitrary objects in $D(X)$ is defined by a stability condition. Thus we follow the original style.

said to be σ -(semi)stable if there is an integer $n \in \mathbb{Z}$ such that $E[n]$ is in \mathcal{A} and $E[n]$ is Z -(semi)stable. We define $\arg Z(E)$ by $\arg Z(E[n]) - n$ and call it the *phase* of E .

We put $\mathcal{P}(\phi) = \{E \in \mathcal{D}(X) | E \text{ is } Z\text{-semistable with phase } \phi\} \cup \{0\}$. Then $\mathcal{P}(\phi)$ is an abelian category. For an interval $I \subset \mathbb{R}$, we define $\mathcal{P}(I)$ by the extension closed full subcategory generated by $\mathcal{P}(\phi)$ for all $\phi \in I$. If for any $\phi \in \mathbb{R}$ there is a positive number ϵ such that $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is artinian and noetherian, then the stability condition $\sigma = (\mathcal{A}, Z)$ is said to be *locally finite*.

In general we cannot define the argument of $Z(E)$ for $E \in D(X)$. However if E is in \mathcal{A} (or $\mathcal{A}[-1]$) then we can define the argument of $Z(E)$ uniquely since the argument $\arg Z(E)$ is in $(0, 1]$ (respectively in $(-1, 0]$).

Take a stability condition $\sigma = (\mathcal{A}, Z)$ on $D(X)$. Then we can easily check that there exists the following sequence of distinguished triangles for an arbitrary object $E \in D(X)$:

$$\begin{array}{ccccccc} 0 & \xrightarrow{\quad} & E_1 & \xrightarrow{\quad} & E_2 & \longrightarrow \cdots \longrightarrow & E_{n-1} & \xrightarrow{\quad} & E_n = E, \\ & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\ & & A_1 & & A_2 & & A_n & & \end{array}$$

[1] [1] [1]

where each A_i is σ -semistable with $\arg Z(A_1) > \cdots > \arg Z(A_n)$. One can easily check that the above sequence is unique up to isomorphism. We also call this sequence the *Harder-Narashimhan filtration* (for short HN filtration). If E is in \mathcal{A} then the above filtration is nothing but the filtration defined in Definition 3.2. In addition assume that σ is locally finite. Then for a σ -semistable object F with phase ϕ we have the following sequence of distinguished triangles:

$$\begin{array}{ccccccc} 0 & \xrightarrow{\quad} & F_1 & \xrightarrow{\quad} & F_2 & \longrightarrow \cdots \longrightarrow & F_{m-1} & \xrightarrow{\quad} & F_m = F, \\ & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\ & & S_1 & & S_2 & & S_m & & \end{array}$$

[1] [1] [1]

where each S_j is σ -stable with $\arg Z(S_j) = \phi$. We call this filtration a *Jordan-Hölder filtration* (for short JH filtration). We remark that a JH filtration of F is not unique but the direct sum $\oplus_{i=1}^m S_i$ of all stable factors of F is unique up to isomorphism.

Now we put

$$\text{Stab}(X) = \{\sigma | \sigma \text{ is a numerical locally finite stability condition on } D(X)\}.$$

Bridgeland [2] describes a subset $U(X)$ of $\text{Stab}(X)$. We shall recall its definition. We put

$$\Delta^+(X) := \{v = r \oplus \delta \oplus s \in \mathcal{N}(X) | v^2 = -2, r > 0\},$$

and define a subset $\mathcal{V}(X)$ of $\text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X)$ by

$$\mathcal{V}(X) := \{(\beta, \omega) \in \text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X) | \langle \exp(\beta + \sqrt{-1}\omega), v \rangle \notin \mathbb{R}_{\leq 0} (\forall v \in \Delta^+(X))\}.$$

Let $(\beta, \omega) \in \mathcal{V}(X)$. Then (β, ω) gives a numerical locally finite stability condition $\sigma_{(\beta, \omega)} = (\mathcal{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$ in the following way. We put $\mathcal{A}_{(\beta, \omega)}$ by

$$\mathcal{A}_{(\beta, \omega)} := \{E^\bullet \in D(X) | H^i(E^\bullet) \begin{cases} \in \mathcal{T}_{(\beta, \omega)} & (i = 0) \\ \in \mathcal{F}_{(\beta, \omega)} & (i = -1) \\ = 0 & (i \neq 0, -1) \end{cases} \},$$

where

$$\begin{aligned} \mathcal{T}_{(\beta, \omega)} &:= \{E \in \text{Coh}(X) | E \text{ is a torsion sheaf or } \mu_{\omega}^-(E/\text{torsion}) > \beta\omega\} \text{ and} \\ \mathcal{F}_{(\beta, \omega)} &:= \{E \in \text{Coh}(X) | E \text{ is torsion free and } \mu_{\omega}^+(E) \leq \beta\omega\}. \end{aligned}$$

We define a stability function $Z_{(\beta, \omega)}$ by $Z_{(\beta, \omega)}(E) := \langle \exp(\beta + \sqrt{-1}\omega), v(E) \rangle$. Then the pair $\sigma_{(\beta, \omega)} = (\mathcal{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$ gives a numerical locally finite stability condition by [2].

Then we put $V(X) := \{\sigma_{(\beta, \omega)} | (\beta, \omega) \in \mathcal{V}(X)\}$. If σ is in $V(X)$ then for any closed point $x \in X$, \mathcal{O}_x is σ -stable with phase 1 by [2, Lemma 6.3]. Let $\tilde{GL}^+(2, \mathbb{R})$ be the universal cover of $GL^+(2, \mathbb{R})$. Then $\text{Stab}(X)$ has the right group action of $\tilde{GL}^+(2, \mathbb{R})$ by [1, Lemma 8.2]. We put $U(X) := V(X) \cdot \tilde{GL}^+(2, \mathbb{R})$. We remark that $U(X)$ is isomorphic to $V(X) \times \tilde{GL}^+(2, \mathbb{R})$.

Let σ be in $\text{Stab}(X)$. Since σ is numerical and the Euler pairing is nondegenerate on $\mathcal{N}(X)$, we have a natural map

$$\pi : \text{Stab}(X) \rightarrow \mathcal{N}(X), \pi(\mathcal{A}, Z) \rightarrow Z^\vee,$$

where $Z(E) = \langle Z^\vee, v(E) \rangle$. The map π gives a complex structure on $\text{Stab}(X)$. In particular each connected component of $\text{Stab}(X)$ is a complex manifold by [1]. If $\pi(\sigma)$ spans a positive real 2-plane and is orthogonal to all (-2) vectors in $\mathcal{N}(X)$ then σ is said to be *good*.

Proposition 3.3. ([2, Proposition 10.3]) *The special locus $U(X)$ is written by*

$$U(X) = \{\sigma \in \text{Stab}(X) | \mathcal{O}_x \text{ is } \sigma\text{-stable with a common phase and } \sigma \text{ is good}\}.$$

Let us consider the boundary $\partial U(X) := \overline{U(X)} \setminus U(X)$ where $\overline{U(X)}$ is the closure of $U(X)$. Then $\partial U(X)$ consists of locally finite union of real codimension 1 submanifolds by [2, Proposition 9.2]. If $\sigma \in \partial U(X)$ lies on only one these submanifold, then σ is said to be *general*.

Theorem 3.4. ([2, Theorem 12.1]) *Let $\sigma \in \partial U(X)$ be general. Then exactly one of the following holds:*

(A^+) : *There is a spherical locally free sheaf A such that both A and $T_A(\mathcal{O}_x)$ are stable factors of \mathcal{O}_x for any $x \in X$, where T_A is the spherical twist by A . Moreover a JH filtration of \mathcal{O}_x is given by*

$$A^{\oplus \text{rank } A} \longrightarrow \mathcal{O}_x \longrightarrow T_A(\mathcal{O}_x) \longrightarrow A^{\oplus \text{rank } A}[1].$$

In particular \mathcal{O}_x is properly σ -semistable⁵ for all $x \in X$ and A does not depend on $x \in X$.

(A^-) : *There is a spherical locally free sheaf A such that both A and $T_A^{-1}(\mathcal{O}_x)$ are stable factors of \mathcal{O}_x for any $x \in X$, where T_A is the spherical twist by A . Moreover a JH filtration of \mathcal{O}_x is given by*

$$T_A^{-1}(\mathcal{O}_x) \longrightarrow \mathcal{O}_x \longrightarrow A^{\oplus \text{rank } A}[2] \longrightarrow T_A^{-1}(\mathcal{O}_x)[1].$$

In particular \mathcal{O}_x is properly σ -semistable for all $x \in X$ and A does not depend on $x \in X$.

(C_k) : *There are a (-2) -curve C and an integer k such that \mathcal{O}_x is σ -stable if $x \notin C$ and \mathcal{O}_x is properly σ -semistable if $x \in C$. Moreover a JH filtration of \mathcal{O}_x for $x \in C$ is given by*

$$\mathcal{O}_C(k+1) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_C(k)[1] \longrightarrow \mathcal{O}_C(k+1)[1].$$

We recall the map $\Phi_* : \text{Stab}(Y) \rightarrow \text{Stab}(X)$ induced by an equivalence $\Phi : D(Y) \rightarrow D(X)$. Let X and Y be projective K3 surfaces, and $\Phi : D(Y) \rightarrow D(X)$ an equivalence. Then Φ induces a natural morphism $\Phi_* : \text{Stab}(Y) \rightarrow \text{Stab}(X)$ as follows:

$$\begin{aligned} \Phi_* : \text{Stab}(Y) &\rightarrow \text{Stab}(X), \quad \Phi_*((\mathcal{A}_Y, Z_Y)) = (\mathcal{A}_X, Z_X) \\ \text{where } Z_X(E) &= Z_Y(\Phi^{-1}(E)), \text{ and } \mathcal{A}_X = \Phi(\mathcal{A}_Y). \end{aligned}$$

Then the following proposition is almost obvious.

⁵Namely \mathcal{O}_x is not σ -stable but σ -semistable.

Proposition 3.5. (*[8, Proposition 6.1]*) *Let X and Y be projective K3 surfaces, and $\Phi : D(Y) \rightarrow D(X)$ an equivalence. For $\sigma \in U(X)$, σ is in $\Phi_*(U(Y))$ if and only if $\Phi(\mathcal{O}_y)$ is σ -stable with the same phase for all closed points $y \in Y$.*

4 Stability of classically stable sheaves

The goal of this section is to show the σ -stability of Gieseker stable (or μ -stable) sheaves on a projective K3 surface X for some $\sigma \in \text{Stab}(X)$.

We first prepare a function (4.1) which plays an important role in this section. Let L_0 be an ample line bundle on X with $L_0^2 = 2d$. We define a subset $V(X)_{L_0}$ of $V(X)$ by

$$V(X)_{L_0} := \{\sigma_{(\beta, \omega)} \in V(X) \mid (\beta, \omega) = (xL_0, yL_0) \text{ where } (x, y) \in \mathbb{R}^2\}.$$

Take an element $\sigma_{(\beta, \omega)} \in V(X)_{L_0}$. For an arbitrary object $F \in D(X)$ we put the Mukai vector $v(F)$ by $v(F) = r_F \oplus \delta_F \oplus s_F$. We have the orthogonal decomposition of δ_F with respect to L_0 in $\text{NS}(X)_{\mathbb{R}}$:

$$\delta_F = n_F L_0 + \nu_F,$$

where ν_F is in $\text{NS}(X)_{\mathbb{R}}$ with $\nu_F L_0 = 0$. Then we have

$$\begin{aligned} Z(F) &= \frac{v(F)^2}{2r_F} + \frac{r_F}{2} \left(\omega + \sqrt{-1} \left(\frac{\delta_F}{r_F} - \beta \right) \right)^2 \\ &= \frac{v(F)^2}{2r_F} + \frac{r_F}{2} \left(\omega + \sqrt{-1} \left(\frac{n_F L_0}{r_F} - \beta \right) \right)^2 - \frac{\nu_F^2}{2r_F}. \end{aligned}$$

We see that the imaginary part $\Im Z(F)$ of $Z(F)$ is given by $2\sqrt{-1}y d\lambda_F$ where $\lambda_F = n_F - r_F x$. Put

$$Z^{L_0}(F) := Z(F) + \frac{\nu_F^2}{2r_F}.$$

We define a function $N_{A,E}(x, y)$ for objects A and $E \in D(X)$ and for $\sigma_{(\beta, \omega)} \in V(X)_{L_0}$ by

$$N_{A,E}(x, y) := \lambda_E \Re Z^{L_0}(A) - \lambda_A \Re Z^{L_0}(E), \quad (4.1)$$

where \Re means taking the real part of a complex number.

Now suppose that E is a μ_{ω} -semistable torsion free sheaf where $\omega \in \text{Amp}(X)$. For a stability condition $\sigma_{(\beta, \omega)} = (\mathcal{A}, Z) \in V(X)$ we see the following:

- $\mu_\omega(E) > \beta\omega \iff E \in \mathcal{A}$.
- $\mu_\omega(E) \leq \beta\omega \iff E \in \mathcal{A}[-1]$.

We shall consider following three cases: $\mu_\omega(E) > \beta\omega$, $\mu_\omega(E) = \beta\omega$ and $\mu_\omega(E) < \beta\omega$. We first treat the case when $\mu_\omega(E) > \beta\omega$.

Lemma 4.1. *Let X be a projective K3 surface and $\sigma_{(\beta,\omega)} = (\mathcal{A}, Z) \in V(X)$. Assume that $A \rightarrow E \rightarrow F$ is a non trivial distinguished triangle in \mathcal{A} . Namely A , E and F are in \mathcal{A} (This means that the triangle gives a short exact sequence in \mathcal{A}).*

(1) *If E is a torsion free sheaf then A is also torsion free sheaf.*

(2) *In addition to (1), if E is Gieseker stable with respect to ω then we have $p_\omega(A) < p_\omega(E)$.*

(3) *Let L_0 be an ample line bundle. In addition to (2), assume $\sigma_{(\beta,\omega)} \in V(X)_{L_0}$ and $\mu_\omega(A) = \mu_\omega(E)$. Then we have $\arg Z(A) < \arg Z(E)$.*

Proof. Let $H^i(F)$ be the i -th cohomology of F . Since F is in \mathcal{A} , $H^i(F)$ is 0 unless i is 0 or -1 . Then one can easily check the first assertion by taking cohomologies to the given distinguished triangle and by this fact. Hence we start the proof of the second assertion (2).

We have the following exact sequence of sheaves:

$$0 \longrightarrow H^{-1}(F) \longrightarrow A \xrightarrow{f} E \longrightarrow H^0(F) \longrightarrow 0.$$

Suppose that $H^{-1}(F)$ is not 0. One can easily see

$$\mu_\omega(H^{-1}(F)) \leq \mu_\omega^+(H^{-1}(F)) \leq \beta\omega < \mu_\omega^-(A) \leq \mu_\omega(A).$$

Thus we have $\mu_\omega(H^{-1}(F)) < \mu_\omega(A) < \mu_\omega(\text{Im } f)$ where $\text{Im } f$ is the image of the morphism $f : A \rightarrow E$. Thus we have $p_\omega(A) < p_\omega(\text{Im } f) \leq p_\omega(E)$ since E is Gieseker stable with respect to ω .

Suppose that $H^{-1}(F) = 0$. Then A is a subsheaf of E . Since E is Gieseker stable, the assertion is obvious.

Let us prove the third assertion. We put $L_0^2 = 2d$. For E and A in $D(X)$ we put $v(E) = r_E \oplus \delta_E \oplus s_E$ and $v(A) = r_A \oplus \delta_A \oplus s_A$. We decompose δ_E and δ_A by

$$\delta_E = n_E L_0 + \nu_E \text{ and } \delta_A = n_A L_0 + \nu_A,$$

where ν_E and ν_A are in $\text{NS}(X)_\mathbb{R}$ with $\nu_E L_0 = \nu_A L_0 = 0$. We remark that both ν_E^2 and ν_A^2 are semi negative and that the number $m_A = \delta_A L_0$ (respectively $m_E = \delta_E L_0$) is an integer. Then we have $n_A = m_A/2d$ (respectively

$n_E = m_E/2d$) and

$$\frac{Z(A)}{r_A} = \frac{v(A)^2}{2r_A^2} + \frac{1}{2} \left(\omega + \sqrt{-1} \left(\frac{n_A L_0}{r_A} - \beta \right) \right)^2 - \frac{\nu_A^2}{2r_A^2}.$$

Now we put $J(A) = \frac{1}{2r_A^2}(v(A)^2 - \nu_A^2)$ and $J(E) = \frac{1}{2r_E^2}(v(E)^2 - \nu_E^2)$. Then we see

$$J(A) = \frac{1}{2} \left(\frac{n_A^2 L_0^2}{r_A^2} - \frac{s_A}{r_A} \right) \text{ and } J(E) = \frac{1}{2} \left(\frac{n_E^2 L_0^2}{r_E^2} - \frac{s_E}{r_E} \right).$$

Since $\mu_\omega(E) = \mu_\omega(A)$ we see $\frac{Z(E)}{r_E} - J(E) = \frac{Z(A)}{r_A} - J(A)$. Thus we see $\arg Z(A) < \arg Z(E)$ if and only if $J(E) < J(A)$. Since $p_\omega(A) < p_\omega(E)$ and $\mu_\omega(A) = \mu_\omega(E)$, we see

$$\frac{n_A}{r_A} = \frac{n_E}{r_E} \text{ and } \frac{s_A}{r_A} < \frac{s_E}{r_E}. \quad (4.2)$$

Then the inequality $J(E) < J(A)$ follows from the inequality (4.2). Thus we have finished the proof. \square

Lemma 4.2. *Let X be a projective K3 surface, let L_0 be an ample line bundle on X and let $\sigma_{(\beta, \omega)} = (\mathcal{A}, Z) \in V(X)_{L_0}$. Assume that $A \rightarrow E \rightarrow F$ is a distinguished triangle in \mathcal{A} with $\text{hom}_X^0(A, A) = 1$ and that E is a torsion free sheaf with $\delta_E = n_E L_0$ for an integer n_E where we put $v(E) = r_E \oplus \delta_E \oplus s_E$.*

(1) *If $v(E)^2 = -2$, $\mu_\omega(A) < \mu_\omega(E)$ and $(\delta_E L_0 - r_E \beta L_0) \leq \frac{\omega^2}{2}$ then $\arg Z(A) < \arg Z(E)$.*

(2) *If $v(E)^2 \geq 0$, $\mu_\omega(A) < \mu_\omega(E)$ and*

$$(\delta_E L_0 - r_E \beta L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \frac{\omega^2}{2}$$

then $\arg Z(A) < \arg Z(E)$.

Proof. We first note that A is a torsion free sheaf by Lemma 4.1. Since there is no $\sigma_{(\beta, \omega)}$ -stable torsion free sheaf with phase 1 (See [8, Remark 3.5 (1)] or [2, Lemma 10.1]), we see $\mu_\omega(A) > \beta\omega$. For the Mukai vector $v(A) = r_A \oplus \delta_A \oplus s_A$ of A we put

$$\delta_A = n_A L_0 + \nu_A,$$

where ν_A is in $\text{NS}(X)_{\mathbb{R}}$ with $\nu_A L_0 = 0$. Then we have

$$\begin{aligned} Z(A) &= \frac{v(A)^2}{2r_A} + \frac{r_A}{2} \left(\omega + \sqrt{-1} \left(\frac{n_A}{r_A} L_0 - r_A \beta \right) \right) - \frac{\nu_A^2}{2r_A} \\ &= Z^{L_0}(A) - \frac{\nu_A^2}{2r_A}. \end{aligned}$$

We note that both $\lambda_A = n_A - r_A x$ and $\lambda_E = n_E - r_E x$ are positive. Since F is in \mathcal{A} , we have $\Im Z(F) \geq 0$. Thus we see $\lambda_A \leq \lambda_E$. Since $\nu_A^2 \leq 0$, we see $\arg Z(A) \leq \arg Z^{L_0}(A)$. Thus it is enough to show that $\arg Z^{L_0}(A) < \arg Z(E)$. Since $\Im Z(A) = \Im Z^{L_0}(A) = 2yd\lambda_A > 0$ and $\Im Z(E) = 2yd\lambda_E > 0$, we see

$$\arg Z^{L_0}(A) < \arg Z(E) \iff 0 < N_{A,E}(x, y).$$

Note that $Z^{L_0}(E) = Z(E)$.

Now we have

$$\begin{aligned} N_{A,E}(x, y) &= \lambda_E \Re Z^{L_0}(A) - \lambda_A \Re Z(E) \\ &= \lambda_E \left(\frac{v(A)^2}{2r_A} + dr_A y^2 - \frac{d\lambda_A^2}{r_A} \right) - \lambda_A \left(\frac{v(E)^2}{2r_E} + dr_E y^2 - \frac{d\lambda_E^2}{r_E} \right) \\ &= dy^2(r_A n_E - r_E n_A) + \lambda_E \frac{v(A)^2}{2r_A} - \lambda_A \frac{v(E)^2}{2r_E} + d\lambda_A \lambda_E \left(\frac{n_E}{r_E} - \frac{n_A}{r_A} \right). \end{aligned}$$

Since $\mu_\omega(A) < \mu_\omega(E)$ we have $\frac{n_E}{r_E} - \frac{n_A}{r_A} > 0$ and $r_A n_E - r_E n_A > 0$. Since the last term $d\lambda_A \lambda_E \left(\frac{n_E}{r_E} - \frac{n_A}{r_A} \right)$ is positive, we have

$$N_{A,E}(x, y) > N'_{A,E}(x, y) := dy^2(r_A n_E - r_E n_A) + \lambda_E \frac{v(A)^2}{2r_A} - \lambda_A \frac{v(E)^2}{2r_E}.$$

Since $\text{hom}_X^0(A, A) = 1$ we have $v(A)^2 \geq -2$. Thus we see

$$N'_{A,E}(x, y) \geq N''_{A,E}(x, y) := dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} - \lambda_A \frac{v(E)^2}{2r_E}.$$

Hence it is enough to prove $N''_{A,E}(x, y) \geq 0$.

Let us prove the first assertion (1). Since $v(E)^2 = -2$ we have

$$\begin{aligned} N''_{A,E}(x, y) &= dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} + \frac{\lambda_A}{r_E} \\ &> dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} \end{aligned}$$

We shall show

$$0 < dy^2(r_An_E - r_E n_A) - \frac{\lambda_E}{r_A}.$$

Since $n_A = m_A/2d$ with m_A and $d \in \mathbb{Z}$, we see

$$\begin{aligned} \frac{\omega^2}{2} = dy^2 &> (\delta_E L_0 - r_E \beta L_0) = 2d\lambda_E \\ &\geq \frac{2d\lambda_E}{r_A(2dr_An_E - r_E m_A)} \\ &= \frac{\lambda_E}{r_A(r_An_E - r_E n_A)}. \end{aligned}$$

Hence we have

$$dy^2(r_An_E - r_E n_A) - \frac{\lambda_E}{r_A} \geq 0.$$

by $r_An_E - r_E n_A > 0$. Thus we have proved the assertion.

Let us prove the second assertion. Essentially the proof is the same as the one of the first assertion. Assume that $v(E)^2 \geq 0$. It is enough to show that $N''_{A,E}(x, y) \geq 0$. Since $0 < \lambda_A \leq \lambda_E$ we have

$$\begin{aligned} N''_{A,E}(x, y) &= dy^2(r_An_E - r_E n_A) - \frac{\lambda_E}{r_A} - \mu_A \frac{v(E)^2}{2r_E} \\ &\geq dy^2(r_An_E - r_E n_A) - \lambda_E - \mu_E \frac{v(E)^2}{2r_E} \end{aligned} \quad (4.3)$$

Hence it is enough to show that $dy^2(r_An_E - r_E n_A) - \lambda_E - \mu_E \frac{v(E)^2}{2r_E} \geq 0$. Similarly to the first assertion, one can easily prove this inequality by using the assumption

$$\frac{\omega^2}{2} \geq (\delta_E L_0 - r_E \beta L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right).$$

Thus we have proved the second assertion. \square

Corollary 4.3. *Notations and assumptions are being as Lemma 4.2. Furthermore we assume that $\text{NS}(X) = \mathbb{Z}L_0$.*

(1) *If $v(E)^2 = -2$, $\mu_\omega(A) < \mu_\omega(E)$ and $\frac{1}{L_0^2}(\delta_E L_0 - r_E \beta L_0) \leq \frac{\omega^2}{2}$, then $\arg Z(A) < \arg Z(E)$.*

(2) *If $v(E)^2 \geq 0$, $\mu_\omega(A) < \mu_\omega(E)$ and*

$$\frac{1}{L_0^2}(\delta_E L_0 - r_E \beta L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \frac{\omega^2}{2},$$

then $\arg Z(A) < \arg Z(E)$.

Proof. We use the same notations as in the proof of Lemma 4.2.

Let us prove the first assertion. Supposet that $v(E)^2 = -2$. By using the same argument in the proof of Lemma 4.2, one can see that it is enough to show that

$$0 \leq dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A}. \quad (4.4)$$

Since $\text{NS}(X) = \mathbb{Z}L_0$, we see $n_A \in \mathbb{Z}$. Thus the inequality (4.4) follows from the assumption $\frac{1}{L_0^2}(\delta_E L_0 - r_E \beta L_0) \leq \frac{\omega^2}{2}$.

One can easily prove the second assertion since the proof is essentially as the same as the first assertion. In fact one can easily see that it is enough to show

$$0 \leq dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} \left(\frac{v(E)^2}{2r_E} + 1 \right), \quad (4.5)$$

instead of (4.4) as above. This inequality follows from the assumption $\frac{1}{L_0^2}(\delta_E L_0 - r_E \beta L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \frac{\omega^2}{2}$. \square

Theorem 4.4. *Let X be a projective K3 surface, L_0 an ample line bundle and $\sigma_{(\beta, \omega)} = (\mathcal{A}, Z) \in V(X)_{L_0}$. We assume that E is a Gieseker stable torsion free sheaf with respect to L_0 with $\mu_\omega(E) > \beta\omega$ and that the Mukai vector $v(E)$ is $r_E \oplus \delta_E \oplus s_E$ with $\delta_E = n_E L_0$ for some $n_E \in \mathbb{Z}$.*

(1) *Assume that $v(E)^2 = -2$. If $\delta_E L_0 - r_E \beta L_0 \leq \omega^2/2$, then E is $\sigma_{(\beta, \omega)}$ -stable.*

(2) *Assume that $v(E)^2 \geq 0$. If $(\delta_E L_0 - r_E \beta L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \omega^2/2$, then E is $\sigma_{(\beta, \omega)}$ -stable.*

Proof. Suppose to the contrary that E is not $\sigma_{(\beta, \omega)}$ -stable. Then there is a $\sigma_{(\beta, \omega)}$ -stable subobject A of E in \mathcal{A} with $\arg Z(A) \geq \arg Z(E)$ and we have the following distinguished triangle in \mathcal{A} :

$$A \longrightarrow E \longrightarrow F \longrightarrow A[1].$$

Since E is Gieseker stable with respect to $\omega = yL_0$ we see that A is a torsion free sheaf with $p_\omega(A) < p_\omega(E)$ by Lemma 4.1. Since $p_\omega(A) < p_\omega(E)$ we see $\mu_\omega(A) \leq \mu_\omega(E)$. If $\mu_\omega(A) = \mu_\omega(E)$ then $\arg Z(A) < \arg Z(E)$ by Lemma 4.1. Thus $\mu_\omega(A)$ should be strictly smaller than $\mu_\omega(E)$. Then whether $v(E)^2 = -2$ or $v(E)^2 \geq 0$, we see $\arg Z(A) < \arg Z(E)$ by Lemma 4.2. Hence E is a $\sigma_{(\beta, \omega)}$ -stable. \square

Corollary 4.5. *Notations and assumptions are being as Theorem 4.4. Furthermore we assume that $\text{NS}(X) = \mathbb{Z}L_0$.*

(1) Assume that $v(E)^2 = -2$. If $\frac{1}{L_0^2}(\delta_E L_0 - r_E \beta L_0) \leq \omega^2/2$, then E is $\sigma_{(\beta, \omega)}$ -stable.

(2) Assume that $v(E)^2 \geq 0$. If $\frac{1}{L_0^2}(\delta_E L_0 - r_E \beta L_0)(\frac{v(E)^2}{2r_E} + 1) \leq \omega^2/2$, then E is $\sigma_{(\beta, \omega)}$ -stable.

The proof of Corollary 4.5 is essentially the same as the proof of Theorem 4.4. The difference is to use Corollary 4.3 instead of Lemma 4.2. Hence we omit the proof. Next we consider the case “ $\mu_\omega(E) = \beta\omega$ ”.

Proposition 4.6. *Let X be a projective K3 surface and $\sigma_{(\beta, \omega)} = (\mathcal{A}, Z) \in V(X)$. Assume that the Mukai vector of an object $E \in D(X)$ is $r_E \oplus \delta_E \oplus s_E$ with $r_E \neq 0$ and $\delta_E \omega / r_E = \beta\omega$.*

(1) *If E is a μ_ω -semistable torsion free sheaf then E is $\sigma_{(\beta, \omega)}$ -semistable with phase 0.*

(2) *The object E is a μ_ω -stable locally free sheaf if and only if E is $\sigma_{(\beta, \omega)}$ -stable with phase 0.*

Proof. Let us prove the first assertion. Since E is μ_ω -semistable, E is in $\mathcal{A}[-1]$. Since $\mu_\omega(E) = \beta\omega$, the imaginary part $\Im Z(E)$ of $Z(E)$ is 0. Thus the argument of $Z(E)$ is 0.

Assume that E is not $\sigma_{(\beta, \omega)}$ -semistable. Then there is a $\sigma_{(\beta, \omega)}$ -semistable object $A \in \mathcal{A}[-1]$ such that

$$A \subset E \text{ in } \mathcal{A}[-1] \text{ with } \arg Z(A) > \arg Z(E) = 0.$$

This contradicts the fact that A is in $\mathcal{A}[-1]$. Hence E is $\sigma_{(\beta, \omega)}$ -semistable.

Let us prove the second assertion. We assume that E is a μ_ω -stable locally free sheaf. Then E is minimal in $\mathcal{A}[-1]$ ⁶ by [5, Theorem 0.2]. Thus E is $\sigma_{(\beta, \omega)}$ -stable with phase 0.

Conversely we assume that E is $\sigma_{(\beta, \omega)}$ -stable with phase 0. Since the rank of E is not 0, E is a locally free sheaf by [2, Lemma 10.1 (b)]. Since E is in $\mathcal{A}[-1]$, we see $E \in \mathcal{F}_{(\beta, \omega)}$. Thus we have

$$\mu_\omega(E) \leq \mu_\omega^+(E) \leq \beta\omega.$$

Thus equalities should hold. Hence E is μ_ω -semistable.

Suppose that E is not μ_ω -stable. Then there is a μ_ω -stable subsheaf A of E such that $\mu_\omega(A) = \mu_\omega(E)$. If necessary by taking a saturation, we may assume that the quotient E/A is a torsion free sheaf. We remark that E/A is μ_ω -semistable. Then A is locally free since E is locally free and $\dim X = 2$.

⁶Namely E has no non-trivial subobject in $\mathcal{A}[-1]$.

Since A is a μ_ω -stable locally free sheaf, A is $\sigma_{(\beta,\omega)}$ -stable with phase 0. Thus the short exact sequence $A \rightarrow E \rightarrow E/A$ defines a distinguished triangle in $\mathcal{A}[-1]$. In particular A is a subobject of E in $\mathcal{A}[-1]$ with phase 0. This contradicts the fact that E is $\sigma_{(\beta,\omega)}$ -stable. \square

Finally we treat the case “ $\mu_\omega(E) < \beta\omega$ ”.

Lemma 4.7. *Let X be a projective K3 surface and $\sigma_{(\beta,\omega)} = (\mathcal{A}, Z) \in V(X)$. Assume that $F \rightarrow E \rightarrow A$ is a distinguished triangle in $\mathcal{A}[-1]$.*

- (1) *If E is a torsion free sheaf then A is a torsion free sheaf.*
- (2) *If E is a μ_ω -stable locally free sheaf then $\mu_\omega(E) < \mu_\omega(A)$.*

We remark that the proof of [8, Lemma 4.4] completely works.

Proof. One can easily prove the first assertion by taking cohomologies to the triangle $F \rightarrow E \rightarrow A$. Thus let us prove the second assertion. Since F , E and A are in $\mathcal{A}[-1]$, we have an exact sequence of sheaves

$$0 \longrightarrow H^0(F) \longrightarrow E \xrightarrow{f} A \longrightarrow H^1(F) \longrightarrow 0,$$

where $H^i(F)$ is the i -th cohomology of F .

Assume that $H^0(F) \neq 0$. Since $H^0(F)$ is torsion free, $\text{rank Im } f < \text{rank } E$, where $\text{Im } f$ is the image of the morphism $f : E \rightarrow A$. Thus $\mu_\omega(E) < \mu_\omega(\text{Im } f)$. By using the fact $H^1(F) \in \mathcal{T}_{(\beta,\omega)}$, one can prove $\mu_\omega(\text{Im } f) \leq \mu_\omega(A)$. Thus we have $\mu_\omega(E) < \mu_\omega(A)$.

Assume that $H^0(F) = 0$. We write F instead of $H^1(F)$. Then E is a subsheaf of A . If $\text{rank } F$ is not 0 then we have $\mu_\omega(A) \leq \beta\omega < \mu_\omega(F)$. Thus we have $\mu_\omega(E) < \mu_\omega(A)$. Suppose that $\text{rank } F = 0$. If the dimension of the support of F is 1 then $c_1(F)\omega > 0$. Hence we see $\mu_\omega(E) < \mu_\omega(A)$. Thus suppose that F is a torsion sheaf with $\dim \text{Supp}(F) = 0$. Take a closed point $x \in \text{Supp}(F)$. By taking the right derived functor $\mathbb{R} \text{Hom}_X(\mathcal{O}_x, -)$ to the triangle $E \rightarrow A \rightarrow F$, we have the following exact sequence of \mathbb{C} -vector spaces:

$$\text{Hom}_X^0(\mathcal{O}_x, E) \rightarrow \text{Hom}_X^0(\mathcal{O}_x, A) \rightarrow \text{Hom}_X^0(\mathcal{O}_x, F) \rightarrow \text{Hom}_X^1(\mathcal{O}_x, E)$$

Since E is locally free we see $\text{Hom}_X^0(\mathcal{O}_x, E) = \text{Hom}_X^1(\mathcal{O}_x, E) = 0$ by the Serre duality. Since x is in the support of F , $\text{Hom}_X^0(\mathcal{O}_x, F)$ should not be 0. This contradicts the torsion freeness of A . Hence we have proved the assertion. \square

Lemma 4.8. *Let X be a projective K3 surface, let L_0 be an ample line bundle, let $\sigma_{(\beta, \omega)} = (\mathcal{A}, Z) \in V(X)_{L_0}$ and let $F \rightarrow E \rightarrow A$ be a distinguished triangle in $\mathcal{A}[-1]$. We put $v(E) = r_E \oplus \delta_E \oplus s_E$. Assume that $\text{hom}_X^0(A, A) = 1$, both $\text{rank } E$ and $\text{rank } A$ are positive and $\delta_E = n_E L_0$ for some integer n_E .*

(1) *If $v(E)^2 = -2$, $\mu_\omega(E) < \mu_\omega(A) < \beta\omega$ and $r_E \beta L_0 - \delta_E L_0 \leq \omega^2/2$, then $\arg Z(E) < \arg Z(A)$.*

(2) *If $v(E)^2 \geq 0$, $\mu_\omega(E) < \mu_\omega(A) < \beta\omega$ and*

$$(r_E \beta L_0 - \delta_E L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \frac{\omega^2}{2},$$

then $\arg Z(E) < \arg Z(A)$.

Proof. The proof is essentially the same as it of Lemma 4.2. We put $L_0^2 = 2d$ and $v(A) = r_A \oplus \delta_A \oplus s_A$ with $\delta_A = n_A L_0 + \nu_A$, where $\nu_A \in \text{NS}(X)_\mathbb{R}$ with $\nu_A L_0 = 0$. If we put $m_A = \delta_A L_0 \in \mathbb{Z}$ then we have $n_A = m_A/2d$.

Now we have

$$\begin{aligned} Z(A) &= \frac{v(A)^2}{2r_A} + \frac{r_A}{2} \left(\omega + \sqrt{-1} \left(\frac{n_A L_0}{r_A} - \beta \right) \right)^2 - \frac{\nu_A^2}{2r_A} \\ &= Z^{L_0}(A) - \frac{\nu_A^2}{2r_A} \end{aligned}$$

Since $\nu_A^2 \leq 0$ we have $\arg Z^{L_0}(A) \leq \arg Z(A)$. Thus it is enough to show that $\arg Z(E) < \arg Z^{L_0}(A)$. We put $\lambda_E = n_E - r_E x$ and $\lambda_A = n_A - r_A x$. We remark that both λ_E and λ_A are negative and $\lambda_E \leq \lambda_A < 0$ by the fact $F \in \mathcal{A}[-1]$. Hence we see

$$\arg Z(E) < \arg Z^{L_0}(A) \iff N_{A,E}(x, y) < 0.$$

Now we have

$$N_{A,E}(x, y) = dy^2(r_A n_E - r_E n_A) + d\lambda_A \lambda_E \left(\frac{n_E}{r_E} - \frac{n_A}{r_A} \right) + \frac{v(A)^2}{2r_A} \lambda_E - \frac{v(E)^2}{2r_E} \lambda_A.$$

Since $\mu_\omega(E) < \mu_\omega(A)$ we see $r_A n_E - r_E n_A < 0$. Thus we have

$$N_{A,E}(x, y) < N'_{A,E}(x, y) := dy^2(r_A n_E - r_E n_A) + \frac{v(A)^2}{2r_A} \lambda_E - \frac{v(E)^2}{2r_E} \lambda_A.$$

Since $\text{hom}_X^0(A, A) = 1$ we have $v(A)^2 \geq -2$. Thus we see

$$N'_{A,E}(x, y) \leq N''_{A,E}(x, y) := dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} - \frac{v(E)^2}{2r_E} \lambda_A.$$

Hence it is enough to show $N''_{A,E}(x, y) \leq 0$.

Assume that $v(E)^2 = -2$, then

$$\begin{aligned} N''_{A,E}(x, y) &= dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} + \frac{\lambda_A}{r_E} \\ &\leq dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} \end{aligned}$$

Hence it is enough to show that

$$dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} \leq 0. \quad (4.6)$$

Recall that $n_A = m_A/2d$ for some integer m_A and $d \in \mathbb{Z}$. Then the inequality (4.6) follows from the assumption

$$r_E \beta L_0 - \delta_E L_0 \leq \frac{\omega^2}{2},$$

Thus we have finished the proof.

Assume that $v(E)^2 \geq 0$. Then we have

$$\begin{aligned} N''_{A,E}(x, y) &= dy^2(r_A n_E - r_E n_A) - \frac{\lambda_E}{r_A} - \frac{v(E)^2}{2r_E} \lambda_A \\ &\leq dy^2(r_A n_E - r_E n_A) - \lambda_E - \frac{v(E)^2}{2r_E} \lambda_E. \end{aligned}$$

Hence it is enough to show that

$$dy^2(r_A n_E - r_E n_A) - \lambda_E - \frac{v(E)^2}{2r_E} \lambda_E \leq 0. \quad (4.7)$$

The inequality (4.7) is equivalent to the following inequality

$$\frac{-\lambda_E}{r_A(r_E n_A - r_A n_E)} \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq dy^2. \quad (4.8)$$

The last inequality (4.8) follows from the assumption

$$(r_E \beta L_0 - \delta_E L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \frac{\omega^2}{2}.$$

Thus we have proved the assertion. \square

Similarly to the case of Corollary 4.5, we have the following corollary. We omit the proof since the proof is as the same as the proof of Lemma 4.8.

Corollary 4.9. *Notations and assumptions are being as Lemma 4.8. Furthermore we assume that $\text{NS}(X) = \mathbb{Z}L_0$.*

(1) *Assume that $v(E)^2 = -2$, $\mu_\omega(E) < \mu_\omega(A) < \beta\omega$ and $\frac{1}{L_0^2}(r_E\beta L_0 - \delta_E L_0) \leq \omega^2/2$. Then we have $\arg Z(E) < \arg Z(A)$.*

(2) *Assume that $v(E)^2 \geq 0$, $\mu_\omega(E) < \mu_\omega(A) < \beta\omega$ and*

$$\frac{1}{L_0^2}(r_E\beta L_0 - \delta_E L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) \leq \omega^2/2.$$

Then we have $\arg Z(E) < \arg Z(A)$.

Theorem 4.10. *Let X be a projective K3 surface, L_0 an ample line bundle and $\sigma_{(\beta,\omega)} = (\mathcal{A}, Z) \in V(X)_{L_0}$. Assume that E is a μ_{L_0} -stable locally free sheaf. We put $v(E) = r_E \oplus \delta_E \oplus s_E$. Assume that $\delta_E = n_E L_0$ and $\mu_\omega(E) < \beta\omega$ where $n_E \in \mathbb{Z}$.*

(1) *Assume that $v(E)^2 = -2$. If $(r_E\beta L_0 - \delta_E L_0) < \omega^2/2$ then E is $\sigma_{(\beta,\omega)}$ -stable.*

(2) *Assume that $v(E)^2 \geq 0$. If $(r_E\beta L_0 - \delta_E L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) < \omega^2/2$ then E is $\sigma_{(\beta,\omega)}$ -stable.*

Proof. Since $\mu_\omega(E) < \beta\omega$, E is in $\mathcal{A}[-1]$ and $\arg Z(E) < 0$. Suppose to the contrary that E is not $\sigma_{(\beta,\omega)}$ -stable. Then there is a $\sigma_{(\beta,\omega)}$ -stable object A such that A is a quotient of E in $\mathcal{A}[-1]$ with $\arg Z(A) \leq \arg Z(E)$. Thus we have a distinguished triangle in $\mathcal{A}[-1]$:

$$F \longrightarrow E \longrightarrow A \longrightarrow F[1].$$

By Lemma 4.7, A is a torsion free sheaf with $\mu_\omega(E) < \mu_\omega(A)$. Since A is in $\mathcal{A}[-1]$, we see $\mu_\omega(A) \leq \beta\omega$. If $\mu_\omega(A) = \beta\omega$, then the imaginary part of $Z(A)$ is 0. Thus A is $\sigma_{(\beta,\omega)}$ -stable with phase 0. This contradicts $\arg Z(A) < \arg Z(E) < 0$. Hence $\mu_\omega(A) < \beta\omega$. Then we see $\arg Z(E) < \arg Z(A)$ by Lemma 4.8 whether $v(E)^2 = -2$ or $v(E)^2 \geq 0$. This is a contradiction. Thus E is $\sigma_{(\beta,\omega)}$ -stable. \square

Corollary 4.11. *Notations and assumptions are being as Theorem 4.10. Furthermore we assume that $\text{NS}(X) = \mathbb{Z}L_0$.*

(1) *Assume that $v(E)^2 = -2$. If $\frac{1}{L_0^2}(r_E\beta L_0 - \delta_E L_0) < \omega^2/2$ then E is $\sigma_{(\beta,\omega)}$ -stable.*

(2) *Assume that $v(E)^2 \geq 0$. If $\frac{1}{L_0^2}(r_E\beta L_0 - \delta_E L_0) \left(\frac{v(E)^2}{2r_E} + 1 \right) < \omega^2/2$ then E is $\sigma_{(\beta,\omega)}$ -stable.*

The proof is essentially as the same as it of Theorem 4.10. One can easily Corollary 4.11 by using Corollary 4.9 instead of 4.8. Hence we omit the proof.

5 First application

The goal of this section is to prove Theorem 5.4 as an application of Corollaries 4.5 and 4.11. We shall give a classification of fine moduli spaces of Gieseker stable torsion free sheaves on a projective K3 surface with Picard number one. In this section the pair (X, L) is called a *generic K3* if X is a projective K3 surface and $\text{NS}(X)$ is generated by an ample line bundle.

We shall start this section with an easy observation. Suppose that E is a Gieseker stable torsion free sheaf on a generic K3 (X, L) . Since E is Gieseker stable we have $v(E)^2 \geq -2$. Assume that $v(E)^2 = -2$. Then $\text{hom}_X^1(E, E) = 0$. Thus E is a spherical sheaf. It is known that E is μ -stable locally free sheaf (For instance see [8, Proposition 5.2]). Thus the notion of μ -stability is equivalent to the notion of Gieseker stability if $v(E)^2 = -2$.

Next we consider the case $v(E)^2 \geq 0$. We write down the following proposition which plays a key roll in this section.

Proposition 5.1. *Let X be a projective K3 surface and L an ample line bundle. Assume that E is a Gieseker stable torsion free sheaf with respect to L with $v(E)^2 = 0$.*

(1) *Assume that $\text{rank } E > 1$. If E is μ -stable with respect to L then E is locally free.*

(2) *Assume that $\text{NS}(X) = \mathbb{Z}L$. If E is locally free then E is μ -stable with respect to L .*

In particular if $\text{NS}(X) = \mathbb{Z}L$ and $\text{rank } E > 1$ then the following holds: If E is not μ -stable locally free then E is neither μ -stable nor locally free.

Proof. The first assertion was proved in the step vii) in the proof of [5, Proposition 4.1].

Hence, let us prove the second assertion. For any $F \in D(X)$ we put $v(F) = r_F \oplus \delta_F \oplus s_F$. Assume that E is not μ -stable. Then there is a μ -stable subsheaf A of E such that $\mu_L(A) = \mu_L(E)$ and the quotient E/A is torsion free. Since E is locally free, A is also locally free. We remark that $p_L(A) < p_L(E)$ since E is Gieseker stable. We remark that $\frac{s_A}{r_A} < \frac{s_E}{r_E}$ by $\mu_L(A) = \mu_L(E)$. Hence we have

$$0 = \frac{v(E)^2}{r_E^2} = \frac{\delta_E^2}{r_E^2} - 2\frac{s_E}{r_E} < \frac{\delta_A^2}{r_A^2} - 2\frac{s_A}{r_A} = \frac{v(A)^2}{r_A^2}.$$

Thus $v(A)^2 > 0$.

We choose $\sigma_{(\beta, \omega)} \in V(X)$ such that $\mu_\omega(E) = \mu_\omega(A) = \beta\omega$. Then E is $\sigma_{(\beta, \omega)}$ -semistable with phase 0 and A is $\sigma_{(\beta, \omega)}$ -stable with phase 0 by Proposition 4.6. Since $\sigma_{(\beta, \omega)}$ is locally finite we have a distinguished triangle

$$A' \longrightarrow E \longrightarrow E/A',$$

where all stable factors of A' are A and $\text{hom}_X^0(A', E/A') = 0$. Then by Lemma 2.3, we see $\text{hom}_X^1(A', A') \leq 2$. Thus $v(A')^2 \leq 0$. However, since A' is an extension of A , we have $v(A') = \ell v(A)$ for some $\ell \in \mathbb{N}$. Thus $v(A')^2 = \ell^2 v(A)^2 > 0$. This is contradiction. Hence E is μ -stable. \square

Suppose that (X, L) is a generic K3 and take an element $v = r \oplus \delta \oplus s \in \mathcal{N}(X)$ with $r > 0$ and $v^2 \geq -2$. We define subsets of $V(X)$ depending on v .

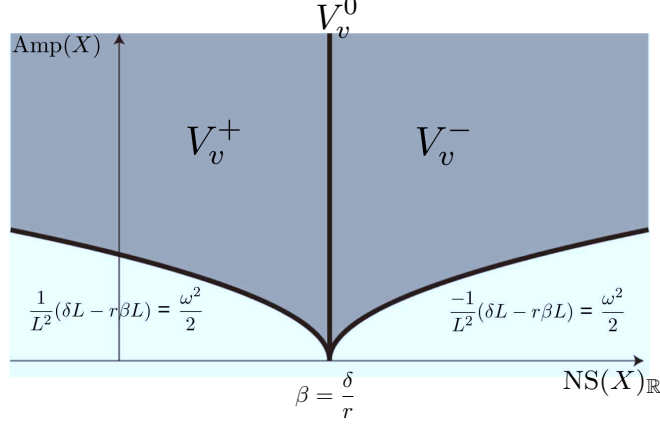
Case 1: $v^2 = -2$.

$$\begin{aligned} V_v^+ &:= \{\sigma_{(\beta, \omega)} \in V(X) | \beta\omega < \frac{\delta\omega}{r}, \frac{1}{L^2}(\delta L - r\beta L) \leq \frac{\omega^2}{2}\}. \\ V_v^0 &:= \{\sigma_{(\beta, \omega)} \in V(X) | \beta\omega = \frac{\delta\omega}{r}\}. \\ V_v^- &:= \{\sigma_{(\beta, \omega)} \in V(X) | \beta\omega > \frac{\delta\omega}{r}, \frac{-1}{L^2}(\delta L - r\beta L) \leq \frac{\omega^2}{2}\}. \end{aligned}$$

Case 2: $v^2 \geq 0$.

$$\begin{aligned} V_v^+ &:= \{\sigma_{(\beta, \omega)} \in V(X) | \beta\omega < \frac{\delta\omega}{r}, \frac{1}{L^2}(\delta L - r\beta L) \left(\frac{v^2}{2r} + 1\right) \leq \frac{\omega^2}{2}\}. \\ V_v^0 &:= \{\sigma_{(\beta, \omega)} \in V(X) | \beta\omega = \frac{\delta\omega}{r}\}. \\ V_v^- &:= \{\sigma_{(\beta, \omega)} \in V(X) | \beta\omega > \frac{\delta\omega}{r}, \frac{-1}{L^2}(\delta L - r\beta L) \left(\frac{v^2}{2r} + 1\right) \leq \frac{\omega^2}{2}\}. \end{aligned}$$

For instance, take a Gieseker stable torsion free sheaf E on (X, L) with $v(E)^2 = 0$ and put $v = v(E) = r \oplus \delta \oplus s$. Then the picture of the sets V_v^+ , V_v^0 and V_v^- are given by the following.



In Proposition 5.2 (below), we show that the set V_v^0 is a “wall” if and only if E is not a μ -stable locally free but Gieseker stable torsion free.

Proposition 5.2. *Let (X, L) be a generic K3 and E a Gieseker stable torsion free sheaf with $v(E)^2 \geq 0$.*

(1) *If the sheaf E is not locally free then E is not σ -semistable for any $\sigma \in V_{v(E)}^-$.*

(2) *If the sheaf E is not μ -stable then E is not σ -semistable for any $\sigma \in V_{v(E)}^-$.*

(3) *Take an arbitrary $\sigma \in V_{v(E)}^-$. For the sheaf E , the following three conditions are equivalent: (a) E is σ -stable, (b) E is σ -semistable and (c) E is μ -stable and locally free.*

Proof. For an object $F \in D(X)$ we put $v(F) = r_F \oplus \delta_F \oplus s_F$. Take an arbitrary element $\sigma_0 = (\mathcal{A}, Z) \in V_{v(E)}^-$.

Let us prove the first assertion (1). Suppose to the contrary that E is σ_0 -semistable. Since E is not locally free, we have the following distinguished triangle by taking double dual of E :

$$S[-1] \longrightarrow E \longrightarrow E^{\vee\vee},$$

where $S = E^{\vee\vee}/E$. Note that S is a torsion sheaf with $\dim \text{Supp}(S) = 0$. Hence $S[-1]$ is σ_0 -semistable with phase 0. Since $\sigma_0 \in V_{v(E)}^-$ we see $\Im m Z(E) < 0$. Hence E is σ -semistable with phase $\phi \in (-1, 0)$. Thus $\arg Z(E) < \arg Z(S[-1])$ and $\text{Hom}_X(S[-1], E)$ should be 0. This contradicts the above triangle. Hence E is not σ_0 -semistable.

Let us prove the second assertion (2). Suppose to the contrary that E is σ_0 -semistable. Since E is not μ -stable, there is a torsion free quotient A of

E such that A is μ -stable with $\mu_L(A) = \mu_L(E)$. Since E is Gieseker stable we have $p_L(E) < p_L(A)$. Thus we see $\frac{s_E}{r_E} < \frac{s_A}{r_A}$. Moreover we can assume that A is locally free. In fact if necessary it is enough to take the double dual of A . Then we see that $\mu_L(A^{\vee\vee}) = \mu_L(E)$, $\frac{s_E}{r_E} < \frac{s_{A^{\vee\vee}}}{r_{A^{\vee\vee}}}$ and $A^{\vee\vee}$ is μ -stable. Thus we can assume that A is a μ -stable locally free sheaf. Note that $\text{Hom}_X^0(E, A) \neq 0$.

We show $V_{v(E)}^- \subset V_{v(A)}^-$. Note that

$$\begin{aligned} r_A \beta L - \delta_A L &= r_A \left(\beta L - \frac{\delta_A}{r_A} L \right) \\ &= r_A \left(\beta L - \frac{\delta_E}{r_E} L \right) \\ &< r_E \left(\beta L - \frac{\delta_E}{r_E} L \right) = r_E \beta L - \delta_E L. \end{aligned} \quad (5.1)$$

Here we use the fact $\text{NS}(X) = \mathbb{Z}L$ in the second inequality.

Since A is μ -stable we have $v(A)^2 \geq -2$. By the definition of $V_{v(A)}^-$, we have to consider two cases. We first assume that $v(A)^2 = -2$. Since $v(E)^2 \geq 0$, we have

$$1 \leq \frac{v(E)^2}{2r_E} + 1.$$

Then we see

$$r_A \beta L - \delta_A L < (r_E \beta L - \delta_E L) \left(\frac{v(E)^2}{2r_E} + 1 \right).$$

Hence we see $V_{v(E)}^- \subset V_{v(A)}^-$ by the definition of $V_{v(E)}^-$.

Next suppose that $v(A)^2 \geq 0$. Then by using the fact that $\text{NS}(X) = \mathbb{Z}L$ we have

$$\begin{aligned} \frac{v(A)^2}{r_A} &= \left(\frac{\delta_A^2}{r_A^2} - 2 \frac{s_A}{r_A} \right) r_A \\ &< \left(\frac{\delta_E^2}{r_E^2} - 2 \frac{s_E}{r_E} \right) r_A \\ &< \left(\frac{\delta_E^2}{r_E^2} - 2 \frac{s_E}{r_E} \right) r_E = \frac{v(E)^2}{r_E}. \end{aligned} \quad (5.2)$$

By two inequalities (5.1) and (5.2) we see

$$(r_A \beta L - \delta_A L) \left(\frac{v(A)^2}{2r_A} + 1 \right) < (r_E \beta L - \delta_E L) \left(\frac{v(E)^2}{2r_E} + 1 \right)$$

Thus we have proved $V_{v(E)}^- \subset V_{v(A)}^-$.

Recall that A is a μ -stable locally free sheaf. Since the stability condition σ_0 is in $V_{v(A)}^-$, A is σ_0 -stable by Corollary 4.11. Now we have

$$\begin{aligned} \frac{Z(A)}{r_A} &= \frac{v(A)^2}{2r_A^2} + \frac{1}{2} \left(\omega + \sqrt{-1} \left(\frac{\delta_A}{r_A} - \beta \right) \right)^2 \\ &= \frac{v(A)^2}{2r_A^2} - \frac{v(E)^2}{2r_E^2} + \frac{v(E)^2}{2r_E^2} + \frac{1}{2} \left(\omega + \sqrt{-1} \left(\frac{\delta_E}{r_E} - \beta \right) \right)^2 \\ &= \frac{Z(E)}{r_E} + \frac{v(A)^2}{2r_A^2} - \frac{v(E)^2}{2r_E^2}. \end{aligned}$$

Here we used the fact $\text{NS}(X) = \mathbb{Z}L$ in the second equality. Since $\mu_L(A) = \mu_L(E)$, $\frac{s_E}{r_E} < \frac{s_A}{r_A}$ and $\text{NS}(X) = \mathbb{Z}L$, we see that $\frac{v(A)^2}{2r_A^2} - \frac{v(E)^2}{2r_E^2}$ is a negative number. Hence we see

$$\arg \frac{Z(A)}{r_A} < \arg \frac{Z(E)}{r_E}.$$

This contradicts $\text{Hom}_X^0(E, A) \neq 0$ since both A and E are σ_0 -semistable. Thus E is not σ_0 -semistable.

Let us prove the third assertion. We claim $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$. The first claim $(a) \Rightarrow (b)$ is trivial. The second claim $(b) \Rightarrow (c)$ follows from the contrapositions of Proposition 5.2 (1) and (2). The third claim $(c) \Rightarrow (a)$ is nothing but Corollary 4.11. Thus we have finished the proof. \square

Take a stability condition $\sigma_{(\beta, \omega)} \in V(X)$ and a μ -semistable torsion free sheaf E with $\mu_\omega(E) = \beta\omega$. By Proposition 4.6, if E is not a μ -stable locally free sheaf, then E is properly σ -semistable. Hence it makes sense to consider a Jordan-Hölder filtration of E with respect to $\sigma_{(\beta, \omega)}$.

Lemma 5.3. *Let X be a projective K3 surface. Take a $\sigma_{(\beta, \omega)} \in V(X)$. Assume that E is a μ_ω -semistable torsion free sheaf with $\mu_\omega(E) = \beta\omega$ and the filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E$$

is a Jordan-Hölder filtration of E with respect to μ_ω -stability. Namely $A_i = E_i/E_{i-1}$ ($i = 1, 2, \dots, n$) is a μ_ω -stable torsion free sheaf with $\mu_\omega(A_i) = \mu_\omega(E)$. Then σ -stable factors of E consist of all $A_i^{\vee\vee}$ and σ -stable factors of $A_i^{\vee\vee}/A_i[-1]$ ($i = 1, 2, \dots, n$).

Proof. We put $\sigma = \sigma_{(\beta, \omega)}$. All A_i ($i = 1, 2, \dots, k$) are σ -semistable by Proposition 4.6. If we obtain JH filtrations of A_i , we can construct a JH filtration of E by combining JH filtrations of A_i . Hence it is enough to prove the assertion for μ -stable torsion free sheaves.

Suppose that A is a μ_ω -stable torsion free sheaf with $\mu_\omega(A) = \beta\omega$ and put $S_A = A^{\vee\vee}/A$. Then we have a distinguished triangle:

$$S_A[-1] \longrightarrow A \longrightarrow A^{\vee\vee} \longrightarrow S_A.$$

Since the dimension of the support of S_A is 0, there are finite closed points $\{x_1, x_2, \dots, x_k\}$ ⁷ such that

$$\begin{array}{ccccccc} 0 & \xrightarrow{\quad} & \mathcal{O}_{x_1} & \xrightarrow{\quad} & F_2 & \longrightarrow \cdots \longrightarrow & F_{k-1} & \xrightarrow{\quad} & F_k = S_A. \\ & \swarrow \scriptstyle [1] & \searrow & \swarrow \scriptstyle [1] & \searrow & & \swarrow \scriptstyle [1] & \searrow & \\ & \mathcal{O}_{x_1} & & \mathcal{O}_{x_2} & & & \mathcal{O}_{x_k} & & \end{array}$$

Since \mathcal{O}_{x_i} ($i = 1, 2, \dots, k$) and $A^{\vee\vee}$ are σ -stable, these are σ -stable factors of A and the JH filtration of A with respect to σ is given by

$$\begin{array}{ccccccc} 0 & \xrightarrow{\quad} & \mathcal{O}_{x_1}[-1] & \xrightarrow{\quad} & F_2[-1] & \longrightarrow \cdots \longrightarrow & F_{k-1}[-1] & \xrightarrow{\quad} & S_A[-1] & \xrightarrow{\quad} & A. \\ & \swarrow \scriptstyle [1] & \searrow & \swarrow \scriptstyle [1] & \searrow & & \swarrow \scriptstyle [1] & \searrow & \mathcal{O}_{x_k}[-1] & \searrow & A^{\vee\vee} \end{array}$$

Thus we have finished the proof. \square

In the next theorem, we give a classification of moduli spaces of Gieseker stable torsion free sheaves on a generic K3 (X, L) . Let Y be the fine moduli space of Gieseker stable torsion free sheaves with Mukai vector $v = r \oplus \delta \oplus s$ and let \mathcal{E} be a universal family of the moduli Y . We define an equivalence $\Phi_{\mathcal{E}} : D(Y) \rightarrow D(X)$ by

$$\Phi_{\mathcal{E}}(-) = \mathbb{R}\pi_{X*}(\mathcal{E} \overset{\mathbb{L}}{\otimes} \pi_Y^*(-)),$$

where π_X (respectively π_Y) is the projection $X \times Y \rightarrow X$ (respectively $X \times Y \rightarrow Y$). To avoid the complexity in notations, we write V^+ (respectively V^0 and V^-) instead of $V_{v(\Phi(\mathcal{O}_y))}^+$ (respectively $V_{v(\Phi(\mathcal{O}_y))}^0$ and $V_{v(\Phi(\mathcal{O}_y))}^-$) for the given equivalence $\Phi_{\mathcal{E}} : D(Y) \rightarrow D(X)$.

Theorem 5.4. *Notations are being as above.*

(1) *If r is not a square number then Y is the fine moduli space of μ -stable locally free sheaf.*

(2) *Assume that r is a square number. Then one of the following two cases occurs:*

⁷There may be i and j in $\{1, 2, \dots, k\}$ so that $x_i = x_j$.

- (a) Y is the fine moduli space of μ -stable locally free sheaves.
- (b) Y is the fine moduli space of properly Gieseker stable torsion free sheaves and Y is isomorphic to X . Moreover $\Phi_{\mathcal{E}}$ is the spherical twist by a spherical locally free sheaf up to an isomorphism $Y \rightarrow X$.

Proof. We note that Y is the fine moduli space of properly Gieseker stable torsion free sheaves or the moduli of μ -stable locally free sheaf by Proposition 5.1. Let $\Phi_{\mathcal{E}*}$ be a natural map $\Phi_{\mathcal{E}*} : \text{Stab}(Y) \rightarrow \text{Stab}(X)$ induced by $\Phi_{\mathcal{E}}$. We put $\mathcal{E}_y = \Phi_{\mathcal{E}}(\mathcal{O}_y)$. Then for any $\sigma \in V^+$, \mathcal{E}_y is σ -stable by Corollary 4.5, and the phase of \mathcal{E}_y does not depend on $y \in Y$. Hence we see $V^+ \subset \Phi_{\mathcal{E}*}U(Y)$. By Proposition 4.6 it is enough to show that $V^0 \cap \Phi_{\mathcal{E}*}U(Y) \neq \emptyset$.

Suppose to the contrary that $V^0 \cap \Phi_{\mathcal{E}*}U(Y) = \emptyset$. We first show that V^0 is contained in the boundary $\partial\Phi_{\mathcal{E}*}U(Y)$ under the assumption $V^0 \cap \Phi_{\mathcal{E}*}U(Y) = \emptyset$. Since V^0 is in the closure of V^+ , V^0 is also in the closure of $\Phi_{\mathcal{E}*}U(Y)$. Then we claim $V^- \cap \Phi_{\mathcal{E}*}U(Y) = \emptyset$. In fact, if $V^- \cap \Phi_{\mathcal{E}*}U(Y) \neq \emptyset$ then \mathcal{E}_y is a μ -stable locally free sheaf for all $y \in Y$ by Proposition 5.2. Moreover V^0 is in $\Phi_{\mathcal{E}*}U(Y)$ by Proposition 4.6. This contradicts $V^0 \cap \Phi_{\mathcal{E}*}U(Y) = \emptyset$. Hence we see $V^- \cap \Phi_{\mathcal{E}*}U(Y) = \emptyset$. Thus V^0 is contained in the boundary $\partial(\Phi_{\mathcal{E}*}U(Y))$. Moreover any $\sigma \in V^0$ is general in $\partial(\Phi_{\mathcal{E}*}U(Y))$ since there are no walls in $V(X)$.

Take a stability condition $\sigma_0 \in V^0$. Recall that the Picard number of X is 1. Since Y is a Fourier-Mukai partner of X , the Picard number of Y is also 1. Since there is no (-2) -curve in Y , \mathcal{O}_y is properly $\Phi_{\mathcal{E}*}^{-1}\sigma_0$ -semistable for all $y \in Y$ by Theorem 3.4. Hence \mathcal{E}_y is not σ_0 -stable but σ_0 -semistable. Moreover we see that \mathcal{E}_y is not a locally free sheaf by Propositions 5.1 and 5.2. Hence we have the following distinguished triangle by taking the double dual of \mathcal{E}_y :

$$S_y[-1] \longrightarrow \mathcal{E}_y \longrightarrow \mathcal{E}_y^{\vee\vee} \longrightarrow S_y,$$

where $S_y = \mathcal{E}_y^{\vee\vee}/\mathcal{E}_y$. By Lemma 2.3, we see

$$\text{hom}_X^1(S_y, S_y) = 2 \text{ and } \text{hom}_X^1(\mathcal{E}_y^{\vee\vee}, \mathcal{E}_y^{\vee\vee}) = 0.$$

Thus there is a closed point $x \in X$ such that $S_y = \mathcal{O}_x$. Since σ_0 is in $V(X)$, \mathcal{O}_x is a σ_0 -stable factor of \mathcal{E}_y . By Theorem 3.4, $\mathcal{E}_y^{\vee\vee}$ is a direct sum of a spherical object S . Since $\mathcal{E}_y^{\vee\vee}$ is a locally free sheaf, S is an also locally free sheaf with $\mu_L(S) = \mu_L(\mathcal{E}_y^{\vee\vee})$. Thus we can put $\mathcal{E}_y^{\vee\vee} = S^{\oplus \ell}$.

Since $v(\mathcal{E}_y) = v(\mathcal{E}_y^{\vee\vee}) - 0 \oplus 0 \oplus 1$, we have

$$0 = v(\mathcal{E}_y)^2 = v(\mathcal{E}_y^{\vee\vee})^2 - 2\langle v(\mathcal{E}_y^{\vee\vee}), v(\mathcal{O}_x) \rangle. \quad (5.3)$$

Furthermore we have $v(\mathcal{E}_y^{\vee\vee})^2 = -2\ell^2$ and

$$\langle v(\mathcal{E}_y^{\vee\vee}), v(\mathcal{O}_x) \rangle = -\text{rank } \mathcal{E}_y^{\vee\vee} = -\text{rank } \mathcal{E}_y = -r.$$

Thus we have

$$2\ell^2 = 2r.$$

Hence if r is not a square number then we have $V^0 \cap \Phi_{\mathcal{E}*}U(Y) \neq \emptyset$. Thus \mathcal{E}_y is a μ -stable locally free sheaf for all $y \in Y$ by Proposition 4.6. This gives the proof of the first assertion (1).

Suppose that $\text{rank } \mathcal{E}_y$ is a square number. Then a JH filtration of \mathcal{E}_y is given by the following triangle:

$$\mathcal{O}_x[-1] \longrightarrow \mathcal{E}_y \longrightarrow S^{\oplus r} \longrightarrow \mathcal{O}_x.$$

Since $\mathcal{O}_x[-1]$ is the unique stable factor of \mathcal{E}_y with an isotropic Mukai vector, one of the following two cases will occur by Theorem 3.4 and by the uniqueness of stable factors up to permutations:

- (i) For any $y \in Y$, there is a closed point $x \in X$ such that $\Phi_{\mathcal{E}} \circ T_B(\mathcal{O}_y) = \mathcal{O}_x[-1]$ where B is a spherical locally free sheaf on Y and T_B is the spherical twist by B .
- (ii) For any $y \in Y$, there is a closed point $x \in X$ such that $\Phi_{\mathcal{E}} \circ T_B^{-1}(\mathcal{O}_y) = \mathcal{O}_x[-1]$ where B is a spherical locally free sheaf on Y .

We remark that B does not depend on y by Theorem 3.4.

Assume that the first case (i) occurs. Then, as is well-known, there is a line bundle M on X and an isomorphism $f : Y \rightarrow X$ such that $\Phi_{\mathcal{E}} \circ T_B(-) = M \otimes f_*(-)[-1]^8$. Thus we have

$$\Phi_{\mathcal{E}}(\mathcal{O}_y) = M \otimes f_*(T_B^{-1}(\mathcal{O}_y))[-1]. \quad (5.4)$$

Then the right hand side of (5.4) is properly complex and the left hand side is a sheaf. This is contradiction. Hence the second case (ii) should occur. Then $\Phi_{\mathcal{E}}(-)$ is given by $M \otimes f_*(T_B(-))[-1]$. This gives the proof of the second assertion (2). \square

Example 5.5. Let (X, L) be a generic K3 and let E be a Gieseker stable torsion free sheaf with $v(E) = r \oplus nL \oplus s$. Since $\text{NS}(X) = \mathbb{Z}L$, E is μ -stable if $\gcd\{r, n\} = 1$ by [3, Lemma 1.2.14]. Then E is a μ -stable locally free sheaf

⁸For instance see [6, Corollary 5.23]

by Proposition 5.1. Moreover if $\gcd\{r, nL^2, s\} = 1$ then the moduli space containing E is a fine moduli space.

Let (X, L) be a generic K3 with $L^2 = 6$. Take $v \in \mathcal{N}(X)$ as $v = 12 \oplus 10L \oplus 25$. Then by [3, Corollary 4.6.7] the moduli space $M_L(v)$ of Gieseker stable torsion free sheaves with Mukai vector v is the fine moduli space since $\gcd\{12, 10L^2, 25\} = 1$. By Theorem 5.4, $M_L(v)$ is the moduli space of μ_L -stable locally free sheaves, although $\gcd\{12, 10\} = 2$.

6 Second application

The goal of this section is to generalize [8, Theorem 1.1] to arbitrary projective K3 surfaces.

In [8] the author describes a picture of $(T_L)_*U(X) \cap V(X)$ by using [8, Theorem 1.2] where T_L is a spherical twist by an ample line bundle L . Instead of the theorem we use Lemma 6.1 (below). Before we state the lemma we prepare the notations. Let X be a projective K3 surface and take an ample line bundle L . For the line bundle L we define the subset $V_L^{>0}$ of $V(X)$ by

$$V_L^{>0} := \{\sigma_{(\beta, \omega)} \in V(X)_L \mid (L^2 - \beta L) \leq \frac{\omega^2}{2}\}.$$

The following lemma is essentially contained in Theorem 4.6. However we write down the lemma to make it much easier to use Theorem 4.6.

Lemma 6.1. *Notations are being as above. The set $V_L^{>0}$ is contained in $T_{L*}U(X)$.*

Proof. Recall that $T_L(\mathcal{O}_x) = L \otimes \mathcal{I}_x[1]$ where \mathcal{I}_x is the kernel of the evaluation map $\mathcal{O}_X \rightarrow \mathcal{O}_x$. If σ is in $V_L^{>0}$ then $L \otimes \mathcal{I}_x$ is σ -stable for all $x \in X$ by Theorem 4.4. Furthermore the phase of $L \otimes \mathcal{I}_x$ does not depend on $x \in X$. Thus we have proved the assertion. \square

The following lemma is also used in [8]. By using Lemma 6.2, we can see $\Phi(\mathcal{O}_y)$ is a sheaf up to shifts if an equivalence $\Phi : D(Y) \rightarrow D(X)$ satisfies the condition $\Phi_*U(Y) = U(X)$.

Lemma 6.2. *([2, Proposition 14.2], [9, Proposition 6.4]) Let X be a projective K3 surface, E in $D(X)$ and $\sigma_{(\beta, \omega)} = (\mathcal{A}, Z) \in V(X)$. We put $v(E) = r_E \oplus \delta_E \oplus s_E$.*

(1) *Assume that $r_E > 0$ and $E \in \mathcal{A}$. If there exists a positive real number ℓ_0 such that E is $\sigma_{(\beta, \ell\omega)}$ -stable for all $\ell > \ell_0$, then E is a torsion free sheaf and is (β, ω) -twisted stable.*

(2) Assume that $r_E = 0$ and $E \in \mathcal{A}$. If there exists a positive real number ℓ_0 such that E is $\sigma_{(\beta, \ell\omega)}$ -stable for all $\ell > \ell_0$, then E is a pure torsion sheaf.

In [8] the author proves that some spherical twists send sheaves to complexes in some special cases. In the following Lemma we generalize this result to arbitrary projective K3 surfaces.

Lemma 6.3. *Let X be a projective K3 surface and let E and A be coherent sheaves with positive rank. We assume that $v(E)^2 = 0$ and $v(A)^2 = -2$ and put $v(E) = r_E \oplus \delta_E \oplus s_E$ and $v(A) = r_A \oplus \delta_A \oplus s_A$.*

(1) *If $(\frac{\delta_E}{r_E} - \frac{\delta_A}{r_A})^2 \geq 0$ then $\chi(A, E) > 0$.*

(2) *In addition to 1, assume that A is spherical and $\text{hom}_X^0(A, E) = 0$. Then the spherical twist $T_A(E)$ of E by A is a complex. In particular the 0-th and 1-st cohomologies survive.*

Proof. We first show the first assertion. Since r_E and r_A are positive, it is enough to show that $\frac{\chi(A, E)}{r_A r_E}$ is positive. We have

$$\begin{aligned} \frac{\chi(A, E)}{r_A r_E} &= -\langle 1 \oplus \frac{\delta_A}{r_A} \oplus \frac{s_A}{r_A}, 1 \oplus \frac{\delta_E}{r_E} \oplus \frac{s_E}{r_E} \rangle \\ &= \frac{s_A}{r_A} + \frac{s_E}{r_E} - \frac{\delta_A \delta_E}{r_A r_E}. \end{aligned}$$

Since $v(A)^2 = -2$ and $v(E)^2 = 0$ we have

$$\frac{s_A}{r_A} = \frac{1}{2} \frac{\delta_A^2}{r_A^2} + \frac{1}{r_A^2} \text{ and } \frac{s_E}{r_E} = \frac{1}{2} \frac{\delta_E^2}{r_E^2}.$$

Thus we have

$$\begin{aligned} \frac{\chi(A, E)}{r_A r_E} &= \frac{1}{2} \frac{\delta_A^2}{r_A^2} + \frac{1}{r_A^2} + \frac{1}{2} \frac{\delta_E^2}{r_E^2} - \frac{\delta_A \delta_E}{r_A r_E} \\ &= \frac{1}{2} \left(\frac{\delta_A}{r_A} - \frac{\delta_E}{r_E} \right)^2 + \frac{1}{r_A^2} > 0. \end{aligned}$$

Thus we have proved the first assertion.

We show the second assertion. By the assumption and (1) of Lemma 6.3 we have $\chi(A, E) = -\text{hom}_X^1(A, E) + \text{hom}_X^2(A, E) > 0$. Hence $\text{hom}_X^2(A, E)$ is not 0. By the computing of the i -th cohomology H^i of $T_A(E)$, we can prove the assertion. In fact we have the following exact sequence of sheaves:

$$\begin{aligned} \text{Hom}_X^0(A, E) \otimes A &\longrightarrow E \longrightarrow H^0 \\ \longrightarrow \text{Hom}_X^1(A, E) \otimes A &\longrightarrow 0 \longrightarrow H^1 \\ \longrightarrow \text{Hom}_X^2(A, E) \otimes A &\longrightarrow 0. \end{aligned}$$

Since $\text{hom}_X^2(A, E)$ is not 0, we see $H^1 \neq 0$. Since $\text{hom}_X^0(A, E)$ is 0, the sheaf H^0 contains E . Thus H^0 is not 0. \square

For an equivalence Φ satisfying the condition $\Phi_{\mathcal{E}*}U(Y) = U(X)$ and for a closed point $y \in Y$, it is enough to prove $\Phi(\mathcal{O}_y) = \mathcal{O}_x[n]$ for some $x \in X$ and $n \in \mathbb{Z}$. By Lemma 6.2, if $\Phi_*U(Y) = U(X)$ then $\Phi(\mathcal{O}_y)$ should be a sheaf up to shifts. Thus we have to exclude the case $\Phi(\mathcal{O}_y)$ is a torsion free sheaf F or pure torsion sheaf T with $\dim \text{Supp}(T) = 1$ (up to shifts). If the Picard number of X is one then it is not necessary to consider the case $\Phi(\mathcal{O}_y) = T$ with $\dim \text{Supp}(T) = 1$ since $v(\Phi(\mathcal{O}_y))^2 = 0$. We need the following lemma to exclude the case $\Phi(\mathcal{O}_y) = T$ with $\dim \text{Supp}(T) = 1$.

Lemma 6.4. *Let X be a projective K3 surface, E a pure torsion sheaf with $\dim \text{Supp}(E) = 1$ and L a line bundle on X . If $\chi(L, E) < 0$ then the spherical twist $T_L(E)$ of E is a sheaf containing a torsion sheaf or a properly complex. In particular $T_L(E)$ is not a torsion free sheaf.*

Proof. Since E is torsion and L is torsion free we have $\text{hom}_X^2(L, E) = 0$ by the Serre duality. Thus we have $\text{hom}_X^1(L, E) \neq 0$ by $\chi(L, E) < 0$. We can compute the i -th cohomology H^i of $T_L(E)$ in the following way:

$$\begin{array}{ccccccc} & & & & 0 & \longrightarrow & H^{-1} \\ & & & & & & \\ & \longrightarrow & \text{Hom}_X^0(L, E) \otimes L & \longrightarrow & E & \longrightarrow & H^0 \\ & & & & & & \\ & \longrightarrow & \text{Hom}_X^1(L, E) \otimes L & \longrightarrow & 0. & & \end{array}$$

Since $\text{hom}_X^1(L, E) \neq 0$ we see $H^0 \neq 0$.

Suppose that $\text{Hom}_X^0(L, E) = 0$. Then $H^{-1} = 0$. We can easily see $H^i = 0$ if $i \neq 0$. Hence $T_L(E)$ is a sheaf containing the torsion sheaf E .

Suppose that $\text{Hom}_X^0(L, E) \neq 0$. Since E is torsion, H^{-1} is not 0. Thus $T_L(E)$ is a complex. \square

In Proposition 6.5 and Corollary 6.6, we generalize [8, Theorem 6.6].

Proposition 6.5. *Let X be a projective K3 surface and E in $D(X)$ with $v(E)^2 = 0$. We put $v(E) = r_E \oplus \delta_E \oplus s_E$.*

(1) *Suppose that $r_E \neq 0$. Then there is a $\sigma \in V(X)$ such that E is not σ -stable.*

(2) *Suppose that $r_E = 0$ and E is σ -stable for all $\sigma \in V(X)$. Then E is $\mathcal{O}_x[n]$ for some closed points $x \in X$ and $n \in \mathbb{Z}$.*

Proof. Let us prove the first assertion (1). Suppose to the contrary that E is σ -stable for all $\sigma \in V(X)$. Since $r_E \neq 0$ we can assume $r_E > 0$ by a shift if necessary. We choose a stability condition $\sigma_{(\beta_0, \omega_0)} = (\mathcal{A}_0, Z_0) \in V(X)$ so that $\frac{\delta_E \omega_0}{r_E} > \beta_0 \omega_0$ and ω_0 is an integral class. Since $\frac{\delta_E \omega_0}{r_E} > \beta_0 \omega_0$ the imaginary part $\Im Z_0(E)$ of $Z_0(E)$ is positive. Hence there is an even integer $2m$ such that $E[2m]$ is in \mathcal{A}_0 . Thus we rewrite E instead of $E[2m]$. Note that E is in \mathcal{A}_0 and r_E is positive.

We consider the following one parameter family of stability conditions

$$\{\sigma_\ell := \sigma_{(\beta_0, \ell \omega_0)} \in V(X) | \ell \in \mathbb{R}_{>0}\}.$$

We put $\sigma_\ell = (\mathcal{A}_\ell, Z_\ell)$. By (1) of Lemma 6.2, E is a (β_0, ω_0) -twisted stable torsion free sheaf.

We choose an ample line bundle L satisfying the following condition:

1. $c_1(L) = n\omega_0$ where n is a positive integer.
2. $\mu_{\omega_0}(L) > \mu_{\omega_0}(E)$.
3. $(\frac{\delta_E}{r_E} - L)^2 > 0$.
4. $r_E - \chi(L, E) < 0$.

This choice is possible if we take a sufficiently large n . Since E is twisted stable, E is μ -semistable with respect to ω_0 . Thus $\text{hom}_X^0(L, E) = 0$ by the second condition for L . Hence $T_L(E)$ is a complex by Lemma 6.3. In particular the 0-th and 1-st cohomologies survive.

Now we put $E' = T_L(E)[1]$ and $v(E') = r' \oplus \delta' \oplus s'$. Since $r' = \chi(L, E) - r_E$, r' is positive. We choose a divisor β so that

$$\beta = bL \ (b \in \mathbb{R}) \text{ and } \beta\omega_0 < \min\{L\omega_0, \frac{\delta' \omega_0}{r'}\}.$$

We consider the following family of stability conditions:

$$\{\sigma_y := \sigma_{(\beta, yL_0)} \in V_L^{>0} | L_0^2 - \beta L_0 \leq \frac{(yL_0)^2}{2}\}.$$

We put $\sigma_y = (Z_y, \mathcal{P}_y)$. By Lemma 6.1, a stability condition σ_y is in $(T_L)_*U(X)$. Since E is τ -stable for all $\tau \in U(X)$, the object E' is $(T_L)_*\tau$ -stable. Thus E' is σ_y -stable since σ_y is in $T_{L*}U(X)$. By the choice of β we have $\Im Z_y(E') > 0$. Hence E' should be a torsion free sheaf up to shifts by (1) of Lemma 6.2. This contradicts the fact that two cohomologies of E' survive.

Let us prove the second assertion (2). We choose an arbitrary stability condition $\sigma_{(\beta_0, \omega_0)} = (\mathcal{A}_0, Z_0) \in V(X)$ and fix it. Since E is $\sigma_{(\beta_0, \omega_0)}$ -stable we can assume that E is in \mathcal{A}_0 by shifts if necessary. By taking a limit $\omega_0 \rightarrow \infty$ we see that E is a pure torsion sheaf by (2) of Lemma 6.2.

We shall show $\delta_E = 0$. Suppose to the contrary that $\delta_E \neq 0$. Then $\delta_E L$ is positive for any ample line bundle L . Thus there is a sufficiently ample line bundle L_0 such that $\chi(L_0, E) < 0$. Here we put $v(T_{L_0}(E)) = r \oplus \delta \oplus s$. Since $r = -\chi(L_0, E)$, we see $r > 0$. Similarly to 1 we consider the following family of stability conditions

$$\{\sigma_y := \sigma_{(0, yL_0)} = (\mathcal{A}_y, Z_y) | L_0^2 \leq \frac{(yL_0)^2}{2}\}.$$

Since $\mu_{L_0}(L_0) = L_0^2 > 0$, σ_y is in $(T_{L_0})_* U(X)$ by Lemma 6.1. Moreover we have

$$\frac{\delta L_0}{r} = \frac{\delta_E - \chi(L_0, E)L_0}{r} L_0 > 0.$$

Thus $\Im m Z_y(T_{L_0}(E)) > 0$. Hence we can assume that $T_{L_0}(E)$ is in \mathcal{A}_y up to even shifts. By (1) of Lemma 6.2 $T_{L_0}(E)$ should be a torsion free sheaf. This contradicts Lemma 6.4. Thus we have $\delta_E = 0$.

Since $\delta_E = 0$, E is a pure torsion sheaf with $\dim \text{Supp}(E) = 0$. Since E is σ -stable we have $\text{hom}_X^0(E, E) = 1$. Thus E is a length 1 torsion sheaf up to shifts. We have proved the assertions. \square

Corollary 6.6. *Let X be a projective K3 surface and E in $D(X)$ with $v(E)^2 = 0$. If E is σ -stable for all $\sigma \in V(X)$ then E is $\mathcal{O}_x[n]$ for some $x \in X$ and $n \in \mathbb{Z}$.*

Proof. We put $v(E) = r_E \oplus \delta_E \oplus s_E$. If $r_E \neq 0$ then this contradicts (1) of Proposition 6.5. Hence $r_E = 0$. The assertion follows from (2) of Proposition 6.5. \square

Theorem 6.7. *Let X and Y be projective K3 surfaces and $\Phi : D(Y) \rightarrow D(X)$ an equivalence. If $\Phi_* U(Y) = U(X)$ then Φ can be written by*

$$\Phi(-) = L \otimes f_*(-)[n]$$

where L is a line bundle on X , f is an isomorphism $f : Y \rightarrow X$ and $n \in \mathbb{Z}$.

Proof. Take an element $\sigma \in \text{Stab}(X)$. By the definition of $\tilde{GL}^+(2, \mathbb{R})$ action we see that an object E is σ -stable if and only if E is $\sigma\tilde{g}$ -stable for all $\tilde{g} \in \tilde{GL}^+(2, \mathbb{R})$. Hence if $\Phi_* U(Y) = U(X)$ then $\Phi(\mathcal{O}_y)$ is written by $\mathcal{O}_x[n]$ for some $x \in X$ and $n \in \mathbb{Z}$ by Corollary 6.6. Then the assertion follows from [6, Corollary 5.23]. \square

Then we immediately obtain the following corollary.

Corollary 6.8. *We put*

$$\mathrm{Aut}(D(X), U(X)) := \{\Phi \in \mathrm{Aut}(D(X)) \mid \Phi_* U(X) = U(X)\}.$$

Then $\mathrm{Aut}(D(X), U(X)) = (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)) \times \mathbb{Z}[1]$.

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